Analysis of a Discrete Complex Sinusoid Frequency Estimator Based on Single-Delay Multiplication Method

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***Abstract*— A statistical analysis of the single-delay multiplication based frequency estimator is treated here. The single-delay multiplication frequency estimator for complex single sinusoid signals is a phase averaged estimator, which is similar to the Kay's estimator. Here we provide a study on the statistical distribution of the frequency estimates made by the estimator, and verify the analytical results using simulations. It is shown that the analytical results hold true even at very low signal to noise ratio levels. In deriving the probability density function of the frequency estimates, we explore the cross product noise terms resulting due to the multiplication operation of the estimator by making valid assumptions.**

I. Introduction

The estimation of frequency of a single tone complex signal is a well treated problem in literature [1-5]. Methods vary from open loop techniques to robust closed loop or feedback techniques. The most attractive estimator in terms of jitter performance is the maximum likelihood (ML) based frequency estimator, which involves high complexity in terms of practical implementation [3-5]. The single-delay multiplication (SD-M) based frequency estimator on the other hand is less efficient with respect to the jitter performance of the ML frequency estimator but with reduced complexity. The single-delay multiplication estimator is a kind of phase averaging estimator, which is similar to the Kay's estimator [1]. Kay, in his paper, provided this technique also with some other modified versions of it by incorporating weighting functions. In this paper we analyse the proposed estimator by performing statistical analysis on it. The statistical distribution of the frequency estimates at the output of the estimator is derived.

II. SINGLE-DELAY MULTIPLICATION BASED FREQUENCY ESTIMATOR

The formation of the estimator is straight forward. Consider the received complex sinusoidal signal corrupted with addictive complex white Gaussian noise,

$$r[n] = A\exp\{j\Omega n + j\theta\} + \eta \tag{1}$$

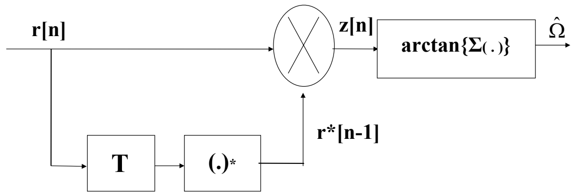

Fig. 1. Block diagram of the single-delay multiplication based frequency estimator.

where, A, Ω and θ are all deterministic constants, and η is the complex white Gaussian noise process given by,

$$\eta = n_c + jn_s \tag{2}$$

where, n_c and n_s are two independent zero mean white Gaussian random processes with σ^2 variance and a double sided power spectral density of N_0 each. Then the single-delay multiplication based frequency estimator is given by,

$$\hat{\Omega} = \arctan\left\{\sum_{n=1}^{L} r[n]r^*[n-1]\right\} \tag{3}$$

where, L+1 is the number of samples required per estimate. For the noiseless case, the estimator in (3) estimates the precise discrete frequency in radians given by,

$$\Omega = 2\pi f T_s \tag{4}$$

where, T_s is the sampling period of the discrete signal and f is the frequency in Hertz. The $\arctan$ function is the four-quadrant inverse tan operation whose statistical properties are treated later in the paper. Fig-1 shows the block diagram of the frequency estimator. Kay's analysis on the frequency estimator, similar to the one given in (3), was based on a linearised expression on the additive noise component for high signal to noise ratio. The model that was used by Kay in [1] is given by,

$$r[n] = A\exp\{j\Omega n + j\theta \ + \nu\} \tag{5}$$

where, ν is a real zero mean white Gaussian random process. This assumption disregards the nonlinear effects caused by the $\arctan$ operation in the estimator. The effect is not apparent at high signal to noise ratio levels, however at low signal to noise ration levels the nonlinear characteristics of the estimator has to be considered in order to provide an appropriate model. Here, in our analysis, we consider the nonlinear characteristics of the $\arctan$ operation and develop a statistical model which holds true even at low signal to noise ratio levels. The analytical results are verified using simulations.

III. Noise Analysis

Consider the block diagram of the estimator given in Fig-1. Let the signal at the output of the multiplier in the figure be $z[n]$. Then, $z[n]$ is given by.

$$z[n] = r[n]r^*[n-1] = A^2[\exp\{j\Omega\} + P_1 + P_2 + \xi] \quad (6)$$

where, the terms, P_1, P_2 and ξ are given by,

$$P_1 = \frac{1}{A}\exp\{j\Omega n + j\theta\}\eta^*[n-1] \quad (7)$$

$$P_2 = \frac{1}{A}\exp\{-j\Omega(n-1) - j\theta\}\eta[n] \quad (8)$$

and

$$\xi = \frac{1}{A^2}\eta[n]\eta^*[n-1] \quad (9)$$

The processes P_1 and P_2, are two linear transforms of the original noise process η, and therefore they follow a Gaussian distribution with the inphase and the quadrature components having zero mean and a variance of σ^2/A^2. Let us now consider the noise cross product term ξ given by (9). The expression in (9) is expanded and rewritten in terms of its inphase and quadrature components as,

$$\xi = \frac{1}{A^2}\{[\xi_1 + \xi_2] + j[\xi_3 - \xi_4]\} \quad (10)$$

where,

$$\xi_1 = n_c[n]n_c[n-1] \quad (11)$$

$$\xi_2 = n_s[n]n_s[n-1] \quad (12)$$

$$\xi_3 = n_s[n]n_c[n-1] \quad (13)$$

$$\xi_4 = n_c[n]n_s[n-1] \quad (14)$$

where, n_c and n_s are the inphase and the quadrature components of the original noise defined in (2). Here, we need to analyse the statistical properties of the terms in (11) and (12). Consider the term ξ_1, since $n_c[n]$ and $n_c[n-1]$ are zero mean Gaussian random variables with variance σ^2, the joint density function of $n_c[n]$ and $n_c[n-1]$ is given by,

$$f_\lambda(\lambda) = \frac{1}{2\pi|\det K|^{1/2}}\exp\{-\frac{1}{2}(\lambda^T K^{-1}\lambda)\} \quad (15)$$

where,

$$\lambda = [n_c[n] \quad n_c[n-1]]^T \quad (16)$$

and, K is the covariance matrix given by,

$$K = \begin{bmatrix} \sigma^2 & \sigma_{11} \\ \sigma_{11} & \sigma^2 \end{bmatrix} \quad (17)$$

In (17), σ_{11} is the covariance of $n_c[n]$ and $n_c[n-1]$, and we recognize that,

$$\sigma_{11} = E[n_c[n]n_c[n-1]] = R_n(T_s) \quad (18)$$

where, $R_n(\tau)$ is the autocorrelation function of n_c at $\tau = T_s$. Assuming that the noise process is ideally low passed filtered with a rectangular brick wall type of filter with a bandwidth of B_i Hz, then the autocorrelation function $R_n(\tau)$ can be expressed as,

$$R_n(\tau) = 2N_0 B_i sinc(2B_i\tau) \quad (19)$$

The autocorrelation function in (19) becomes zero when the filter bandwidth B_i satisfies the condition,

$$B_i = \frac{m}{2T_s} \quad (20)$$

where, m is a positive integer. For discrete systems, when signals are ideally sampled at a sampling frequency of $1/T_s$ Hz, equation (20) holds true with $m = 1$. Thus, when equation (20) is satisfied, the covariance of $n_c[n]$ and $n_c[n-1]$ becomes zero, resulting in a joint distribution function given by,

$$f_\lambda(\lambda) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{n_c^2[n] + n_c^2[n-1]}{2\sigma^2}\right\} \quad (21)$$

Now, by using the result in (21) and making the necessary linear transformation on $n_c[n]$, or otherwise, we can find the probability density function of ξ_1, given by,

$$f_{\xi_1}(\xi_1) = \int_{-\infty}^{\infty}\frac{1}{2\pi|\beta|\sigma^2}\exp\left\{-\frac{\beta^2 + \xi_1^2/\beta^2}{2\sigma^2}\right\}d\beta \quad (22)$$

where, $\beta = n_c[n]$. A closed form expression for the probability density function in (22) is not explicit. Hence, numerical techniques were used to solve the integral in (22). The results were verified using simulations and are shown in Fig-2. Even though a closed form expression for the pdf was not obtained in (22), the mean and the variance of ξ_1 are easily obtained by direct analysis. The variable ξ_1 has zero mean because of the symmetrical nature of the pdf in (22). The variance is computed by solving a double integral involving (22), and is given by σ^4/A^4. With these first order statistics of ξ_1, we can determine the same for the real and the imaginary components of ξ summed over L terms, given in (10). Let,

$$\chi = \sum_{n=1}^{L}[\xi_1 + \xi_2] \quad (23)$$

Note that ξ_1 and ξ_2 are intrinsically functions of n as described in (11) and (12) respectively. Since the variances of ξ_1 and ξ_2 are bounded, using the central limit theorem [7], we find that the distribution of χ is a Gaussian for large values of L (typically for $L > 5$ from simulations). The mean and the variance of χ are given by $m_\chi = 0$ and $\sigma_\chi^2 = 2L\sigma^4/A^4$,

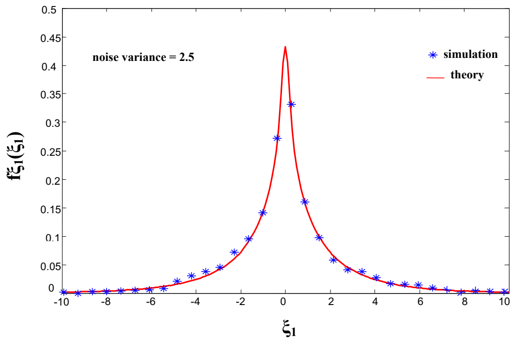

Fig. 2. PDF of the noise product term $\xi_1, A = 1, \sigma^2 = 2.5$.

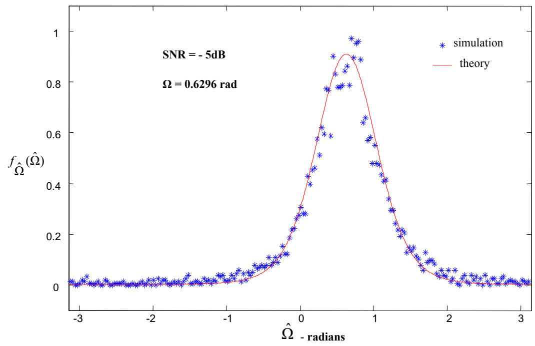

Fig. 4. PDF of the frequency estimates $\hat{\Omega}$, SD-M estimator

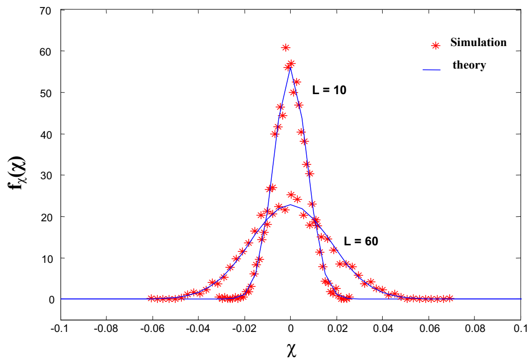

Fig. 3. Simulation and analytical results for the PDF of χ.

respectively for L > 5. The above results are verified by simulation and are shown in Fig-3 for L = 10 and L = 60. Now using the noise analysis performed in this section, we determine the distribution of the frequency estimates made by the SD-M estimator in the following section.

IV. STATISTICAL DISTRIBUTION OF THE FREQUENCY ESTIMATES

The statistical properties of the frequency estimates made at the output of the SD-M estimator are of great interest to us. Especially one needs to know how the estimates behave statistically. Let us rewrite the expression in (3) taking into account the additive noise component.

$$Z_L = \sum_{n=1}^{L} z[n] = A^2\{L\exp\{j\Omega\} + \chi_1 + \chi_2 + \chi_3\} \quad (24)$$

where,

$$\chi_1 = \sum_{n=1}^{L} P_1 \ , \chi_2 = \sum_{n=1}^{L} P_2 \ and \ \chi_3 = \sum_{n=1}^{L} \xi \quad (25)$$

In the above equation, χ_1 and χ_2 are Gaussian distributed random processes, each with means zero and variances $L\sigma^2/A^2$. The real and the imaginary components of χ_3 have similar statistical properties as χ. Now, we could rewrite (24) as,

$$Z_L = Z_1 + jZ_2 \quad (26)$$

In (26), Z_1 and Z_2 are two Gaussian random processes with means μ_1 and μ_2 respectively, and variances σ_2^2 each, given by,

$$\mu_1 = A^2 L\cos(\Omega) \ , \mu_2 = A^2 L\sin(\Omega) \quad (27)$$

and

$$\sigma_2^2 = 2L\sigma^2 + 2L\frac{\sigma^4}{A^4} \quad (28)$$

Then, the frequency estimate at the output of the SD-M estimator is given by the four-quadrant $\arctan$ function of Z_L,

$$\hat{\Omega} = \arctan\{\frac{Z_2}{Z_1}\} \quad (29)$$

The equation in (29) is treated as a transformation of multiple random variables in order to determine the output distribution. Since the distributions of Z_1 and Z_2 are known, we can determine the statistical distribution of the frequency estimates made at the output of the estimator, which is given by,

$$f_{\hat{\Omega}}(\hat{\Omega}) = \int_{-\infty}^{\infty} f_{z_1 z_2}(z_1, z_2)|J(z_1, z_2)|^{-1} dz_1 \quad (30)$$

where, $z_2 = z_1\tan(\hat{\Omega})$, and $|J(z_1, z_2)|$ is the determinant of the Jacobian matrix for the transformation given in (29), which is given by,

$$|J(z_1, z_2)| = \begin{vmatrix} \frac{\partial z_1}{\partial \varphi} & \frac{\partial z_1}{\partial z_2} \\ 0 & 1 \end{vmatrix}^{-1} \quad (31)$$

The probability density function of $\hat{\Omega}$ is then given by solving the integral in (30) and the results are given in (32), (33) and (34). Since the trigonometric $\arctan$ function has got two discontinuities within the range of $-\pi$ to π , one at $-\pi/2$ and the other at $\pi/2$, the integral in (30) needs to be evaluated for three different regions. The analytical results obtained for the statistical distribution of the frequency estimate at the output of the SD-M estimator is verified using simulations, and the results are shown in Fig-4. The proposed statistical model for the estimator matches very closely to the simulated results.

$$f_{\hat{\Omega}}(\hat{\Omega}) = \begin{cases} \exp(-\varsigma/2)\left[\frac{1}{2\pi} - \exp\left(\frac{\Lambda_1}{2}\right)\left(\frac{\Lambda_1}{2\pi}\right)^{\frac{1}{2}} Q\left(\sqrt{\Lambda_1}\right)\right] & \text{for } \pi/2 < \hat{\Omega} \le \pi \\ \exp(-\varsigma/2)\left[\frac{1}{2\pi} + \exp\left(\frac{\Lambda}{2}\right)\left(\frac{\Lambda}{2\pi}\right)^{\frac{1}{2}} \left(1 - Q\left(\sqrt{\Lambda}\right)\right)\right] & \text{for } -\pi/2 < \hat{\Omega} \le \pi/2 \\ \exp(-\varsigma/2)\left[\frac{1}{2\pi} - \exp\left(\frac{\Lambda_2}{2}\right)\left(\frac{\Lambda_2}{2\pi}\right)^{\frac{1}{2}} Q\left(\sqrt{\Lambda_2}\right)\right] & \text{for } -\pi < \hat{\Omega} \le -\pi/2 \end{cases} \tag{32}$$

$$where\ , \varsigma = \frac{\mu_1^2 + \mu_2^2}{\sigma_2^2}\ \Lambda = \frac{\left[\tan\left(\hat{\Omega}\right)\mu_1 + \mu_2\right]^2}{\left[1 + \tan^2\left(\hat{\Omega}\right)\right]\sigma_2^2}\ \Lambda_1 = \frac{\left[\tan\left(\pi + \hat{\Omega}\right)\mu_1 + \mu_2\right]^2}{\left[1 + \tan^2\left(\pi + \hat{\Omega}\right)\right]\sigma_2^2} \tag{33}$$

$$\Lambda_2 = \frac{\left[\tan\left(-\pi + \hat{\Omega}\right)\mu_1 + \mu_2\right]^2}{\left[1 + \tan^2\left(-\pi + \hat{\Omega}\right)\right]\sigma_2^2}\ and\ , Q(x) = \frac{1}{2\pi}\int_x^{\infty} \exp\left(-u^2/2\right) du \tag{34}$$

V. Conclusion

Statistical analysis of the single-delay multiplication based frequency estimator was treated here. The distribution of the frequency estimates made at the output of the estimator was determined and verified using simulations. The statistical model is very precise even at very low signal to noise ratio levels, and is of great use in receiver designs to estimate the jitter and the behavior of the estimator when operating under low signal to noise ratio conditions.

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